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Anomalous Behavior of a Scalar Particle in a Globally Regular Space-Time of a Schwarzschild Black Hole

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The curved space-time Klein-Gordon equation in a globally regular space-time of a Schwarzschild black hole is solved, and its exact solution is obtained. The wave functions of a scalar particle inside the black hole are discussed by means of numerical analysis. The anomalous behaviors of the scalar particle in the central region of the black hole and in the interior neighborhood of the Schwarzschild event horizon are studied with the help of approximate solutions, which are compared with the exact one in these two regions.

1. INTRODUCTION

Recently, great attention has been paid to the globally regular space-time metric of a black hole (Shen and Zhu, 1985) and the behavior of a particle outside and inside a Schwarzschild black hole (Zhu and Shen, 1986). Gonzales-Diaz (1981) has presented the space-time metric inside the Schwarzschild black hole. The metric has the two most interesting features of strong attractions: an ultraviolet free-field asymptotic behavior for $r = 0$ and an infrared divergence for the event horizon $r = R_s$. A new concept of the globally regular solution of the Einstein field equations has recently been presented (Shen and Zhu, 1985), namely, that the solutions of the Einstein field equations which are free from singularities and satisfy the dominant energy condition are the globally regular solutions. An equation of state not satisfying the strong energy condition (Hawking and Ellis, 1973) can still be considered as physically reasonable. For instance, the equation of state $P = -\rho$ may describe mesons in the relativistic dense state (Hakim,

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1978). Kofinti (1984) has calculated the form of the wave function of a scalar particle in the exterior neighborhood of the Schwarzschild event horizon. Although the radial wave function of a scalar particle is accompanied by an infinite phase factor, its probability density remains finite. Hence, the event horizon is asserted to be physically nonsingular. The study of the geodesic motion of a test particle has shown that an infinite potential barrier occurs at the center of the black hole for a particle with nonzero angular momentum, and that there exists a classical trivial solution $r=0$ for a particle with zero angular momentum (Zhu and Shen, 1986). In view of the globally regular solution for a black hole with homogeneous body distributions, it is interesting to investigate further the characteristics of a scalar particle in the space-time with this kind of metric.

This paper is devoted to the study of the behavior of a scalar particle in globally regular space-time of a Schwarzschild black hole. The curved space-time Klein-Gordon equation is solved, and its exact solution in the whole interior of a Schwarzschild black hole is found. The wave function and the corresponding probability density of a scalar particle inside the black hole are discussed with the aid of numerical analysis. The explicit approximate expressions for the radial wave functions in the interior neighborhood of the event horizon and in the central region of the black hole are obtained so as to reveal clearly the anomalous behavior of a scalar particle in these two regions. The results predicted by these expressions are compared with the exact solution.

2. KLEIN–GORDON WAVE EQUATION IN THE SCHWARZSCHILD INTERIOR SPACE-TIME

The curved space-time Klein–Gordon wave equation for a scalar particle of rest mass μ can be written as

$$\left[\frac{1}{(-g)^{1/2}}\frac{\partial}{(\partial x^{\mu}}\left((-g)^{1/2}\, g^{\mu\nu}\frac{\partial}{\partial x^{\nu}}\right)+\mu^2\right]\Phi(t, x_1, x_2, x_3)=0 \qquad (1)$$

in units $\hbar=c=G=1$, where $\Phi\ (t, x_1, x_2, x_3)$ is the wave function of the particle.

A Schwarzschild black hole with mass M has the interior metric given by (Shen and Zhu, 1985)

$$ds^2=(1-r^2/R_s^2)\, dt^2-(1-r^2/R_s^2)^{-1}\, dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2) \qquad (2)$$

in spherical polar coordinates (t, r, θ, ϕ), where $R_s=2M$ is the radius of the Schwarzschild black hole.

In the metric (2), the wave equation (1) takes the form

$$\left[\left(1-\frac{r^2}{R_s^2}\right)^{-1}\frac{\partial^2}{\partial t^2}-\left(1-\frac{r^2}{R_s^2}\right)\frac{\partial^2}{\partial r^2}-\frac{2}{r}\left(1-2\frac{r^2}{R_s^2}\right)\frac{\partial}{\partial r}\right.$$
$$\left.-\frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2}+ctg\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)+\mu^2\right]\Phi=0 \tag{3}$$

If the wave function $\Phi(t, r, \theta, \phi))$ can be expressed as

$$\Phi(t, r, \theta, \phi)=T(t)R(r)\Theta(\theta, \phi)$$

then equation (3) can be separated into the following three equations:

$$\frac{d^2T(t)}{dt^2}+w^2T(t)=0 \tag{4}$$

$$\left(1-\frac{r^2}{R_s^2}\right)^2\frac{d^2R_l}{dr^2}+\frac{2}{r}\left(1-\frac{r^2}{R_s^2}\right)\left(1-\frac{2r^2}{R_s^2}\right)\frac{dR_l}{dr}$$
$$+\left\{w^2-\left(1-\frac{r^2}{R_s^2}\right)\left[\mu^2+\frac{l(l+1)}{r^2}\right]\right\}R_L=0 \tag{5}$$

$$\left[\frac{\partial^2}{\partial\theta^2}+ctg\,\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}+L(L+1)\right]\Theta_{Lm}=0 \tag{6}$$

where w is a separation constant, corresponding to the frequency of the wave, and L is a nonnegative integer, relating to the orbital angular momentum quantum number of the scalar particle. Equations (4) and (6) have the solutions

$$T(t)=a\,e^{-iwt}+b\,e^{iwt} \tag{7}$$

$$\Theta_{Lm}(\theta, \phi)=Y_L^m(\cos\theta)\exp(im\phi) \tag{8}$$

where a and b are arbitrary constants, Y_L^m $(\cos\theta)$ are spherical harmonics, and m is the magnetic quantum number, an integer such that $|m|\leq L$. Hence the wave equation (3) has the eigensolutions

$$\Phi(t, r, \theta, \phi)=NR_L(r)Y_L^m(\cos\theta)\exp[i(m\phi\pm wt)]$$

where $R_L(r)$ is a solution of the radial wave equation (5), and N is a normalization factor.

The probability current density of a scalar particle can be defined as

$$J_\mu=-\frac{i}{2\mu}\left(\Phi^*\frac{\partial}{\partial x^\mu}\Phi-\Phi\frac{\partial}{\partial x^\mu}\Phi^*\right) \tag{9}$$

to satisfy the conservation equation

$$\partial_{;\mu}J^\mu=0 \tag{10}$$

where $\partial_{;\mu}$ denotes the covariant derivative in the curved space-time. Only the positive frequency term in (7) will be retained, so as to avoid negative probability problems. Then the probability density is

$$P = J_0 = -\frac{i}{2\mu}[iW\Phi^*\Phi - \Phi\Phi^*(-iw)] = \frac{W}{\mu}|\Phi|^2 \geq 0$$

3. EXACT RADIAL SOLUTION INSIDE THE SCHWARZSCHILD BLACK HOLE

The radial equation (5) can be rewritten as

$$\frac{1}{r^2}\left(1-\frac{r^2}{R_s^2}\right)\frac{d}{dr}\left[r^2\left(1-\frac{r^2}{R_s^2}\right)\frac{dR_L}{dr}\right]+\left\{w^2-\left(1-\frac{r^2}{R_s^2}\right)\left[\mu^2+\frac{L(L+1)}{r^2}\right]\right\}R_L=0 \tag{11}$$

By putting

$$R_L(r) = U_L(r)/r$$

the radial equation (11) becomes

$$\left(1-\frac{r^2}{R_s^2}\right)\frac{d}{dr}\left[\left(1-\frac{r^2}{R_s^2}\right)\frac{dU_L}{dr}\right]+\left\{w^2-\left(1-\frac{r^2}{R_s^2}\right)\left[\mu^2+\frac{L(L+1)}{r^2}-\frac{2}{R_s^2}\right]\right\}U_L = 0 \tag{12}$$

Finally, the substitution

$$x = r/R_s \qquad (0 \leq x \leq 1)$$

reduces equation (12) to the form

$$(1-x^2)\frac{d}{dx}\left[(1-x^2)\frac{dU_L}{dx}\right]+\left\{\omega^2-(1-x^2)\left[\frac{L(L+1)}{x^2}-p\right]\right\}U_L=0 \tag{13}$$

where $\omega = wR_s$ and $p = (2-\mu^2R_s^2)$.

A inspection shows that the solution of equation (13) can be expanded in a series

$$U_L(x) = a_s x^s + a_{s+1}x^{s+1} + \cdots + a_{s+k}x^{s+k} + \cdots \tag{14}$$

with $a_s \neq 0$. Inserting equation (14) into (13) yields the set of equations

$$S(S-1)a_s = L(L+1)a_s \tag{15}$$

$$(S+1)Sa_{s+1} = L(L+1)a_{s+1} \tag{16}$$

$$(S+2)(S+1)a_{s+2} - (S+1)Sa_s - S(S-1)a_s = L(L+1)(a_{s+2}-a_s) - \omega^2 a_s - pa_s \tag{17}$$

$$
\begin{aligned}
&(S+3)(S+2)a_{s+3}-(S+2)(S+1)a_{s+1}-(S+1)Sa_{s+1}\\
&\quad = L(L+1)(a_{s+3}-a_{s+1})-\omega^2 a_{s+1}-pa_{s+1}
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
&\vdots\\
&(S+k+2)(S+k+1)a_{s+k+2}-(S+k+1)(S+k)a_{s+k}\\
&\qquad -(S+k)(S+k-1)a_{s+k}+(S+k-1)(S+k-2)a_{s+k-2}\\
&\quad = L(L+1)(a_{s+k+2}-a_{s+k})-\omega^2 a_{s+k}-p(a_{s+k}-a_{s+k-2})\\
&\vdots
\end{aligned} \tag{19}
$$

It is easy to obtain two roots from equation (15), that is,

$$S_1 = L+1 \tag{20}$$

and

$$S_2 = -L \tag{21}$$

together with another solution from equation (16)

$$a_{s+1}=0 \tag{22}$$

It may be proved that the coefficients satisfy the following recurrence relations:

$$
\begin{aligned}
a_{s+2m} &= \frac{2(s+2m-2)^2-L(L+1)-\omega^2-p}{(s+2m)(s+2m-1)-L(L+1)}\,a_{s+2m-2}\\
&\quad +\frac{p-(s+2m-3)(s+2m-4)}{(s+2m)(s+2m-1)-L(L+1)}\,a_{s+2m-4}\\
a_{s+2m-3} &= 0\\
a_{s+2} &= \frac{2s^2-L(L+1)-\omega^2-p}{(s+2)(s+1)-L(L+1)}\,a_s
\end{aligned} \tag{23}
$$

where $m=2,3,4,\ldots$ and $a_s \neq 0$. Since

$$R=\lim_{m\to\infty}|a_{s+2m-2}/a_{s+2m}|=1 \tag{24}$$

the solution (14) of equation (13) is analytic in the region $0\leq x\leq 1$.

If we require that the radial wave functions are square integrable in the central region of the black hole, then the solution of equation (11) can be written as

$$
R_L(r)=\begin{cases}
\dfrac{c_0}{r}\,Dh^{(1)}_{\omega 0}\left(\dfrac{r}{R_s}\right)+\dfrac{d_0}{r}\,Dh^{(2)}_{\omega 0}\left(\dfrac{r}{R_s}\right) & \text{for}\quad L=0 \qquad (25)\\[2ex]
\dfrac{C_L}{r}\,Dh^{(1)}_{\omega L}\left(\dfrac{r}{R_s}\right) & \text{for}\quad L\geq 1 \qquad (26)
\end{cases}
$$

with

$$Dh^{(1)}_{\omega L}(x) = x^{L+1}\left(\sum_{m=0}^{\infty} a_{L+1+2m}x^{2m}\right) \tag{27}$$

and

$$Dh^{(2)}_{\omega L}(x) = x^{-L}\left(\sum_{m=0}^{\infty} b_{2m-L}x^{2m}\right) \tag{28}$$

The coefficients in equations (27) and (28) satisfy the recurrence relations

$$a_{L+1+2m} = \frac{2(L+1+2m)^2 - L(L+1) - \omega^2 - p}{(2m+L+1)(2m+L) - L(L+1)} a_{L+2m-1} - \frac{(L+2m-2)(L+2m-3) - p}{(L+2m+1)(L+2m) - L(L+1)} a_{L+2m-3} \tag{29}$$

and

$$b_{2m-L} = \frac{2(2m-L-2)^2 - L(L+1) - \omega^2 - p}{(2m-L)(2m-L-1) - L(L+1)} b_{2m-L-2} - \frac{(2m-L-3)(2m-L-4) - p}{(2m-L)(2m-L-1) - L(L+1)} b_{2m-L-4} \tag{30}$$

with $a_k = 0$ for $k < L+1$ and $b_n = 0$ for $n < -L$.

In order to gain an insight into the behavior of a scalar particle inside the Schwarzschild black hole, we perform numerical analysis of the functions $Dh^{(1)}_{\omega L}(x)$ and $Dh^{(2)}_{\omega L}(x)$. Figure 1 shows $Dh^{(1)}_{\omega L}(x)$ for $a_{L+1} = 1$, $p = 1.50$ and $\omega = 10$ and different values of L. Figure 2 shows $Dh^{(2)}_{\omega L}(x)$ for $a_{-L} = 1$, $p = 1.50$, $\omega = 10$, and $L = 0$, 1.

As can be seen from Figures 1 and 2, $Dh^{(1)}_{\omega L}(x)$ and $Dh^{(2)}_{\omega L}(x)$ are analytic functions of x inside the black hole including the event horizon. Besides, if $Dh^{(1)}_{\omega L}(x)$ and $Dh^{(2)}_{\omega L}(x)$ are considered as functions of ω and L, they are also analytic for all possible values of ω and L.

Now we can see that for a particle with zero orbital angular momentum an infinite potential pit occurs in the central region of the black hole. However, $R_L(r)$ for $L \geq 1$ tends to zero when $r \to 0$, and consequently a particle with nonzero orbital angular momentum cannot reach the center. The distance of the first maximum apart from the origin increases with the increase of L. If L is large enough, the radial wave functions will attenuate in an exponential way.

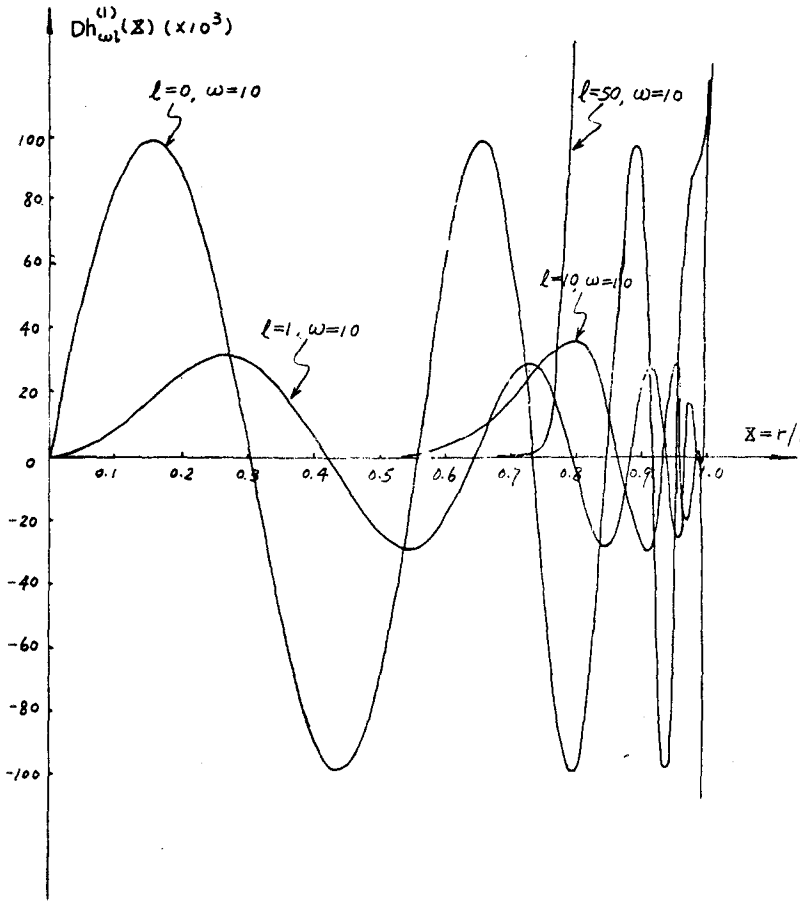

Fig. 1. Plot of $Dh_{\omega L}^{(1)}(x)$ as a function of x for $a_{L+1}=1$, $p=1.50$, $\omega=10$, and different values of L.

Moreover, as $r \to R_s$, $|R_L(r)|$ remains finite. Hence the radial probability density at $r=R_s$ is limited, and the event horizon is physically nonsingular. The wave function with relatively high frequency, which is accompanied by a divergent phase factor near the event horizon, appears as a rapidly oscillating state. Furthermore, the probability density reduces all at once within a very small range near the event horizon. Therefore, the state of a scalar particle is localized around the event horizon. The scalar particle with larger L is localized in the narrower region.

$Dh_{\omega L}^{(1)}(x)$ as a function of x is calculated for $a_{L+1}=1$, $p=1.50$, $L=1$, and ω ranging from 0 to 2.0, and two graphs for $\omega=0$ and 2.0 are shown

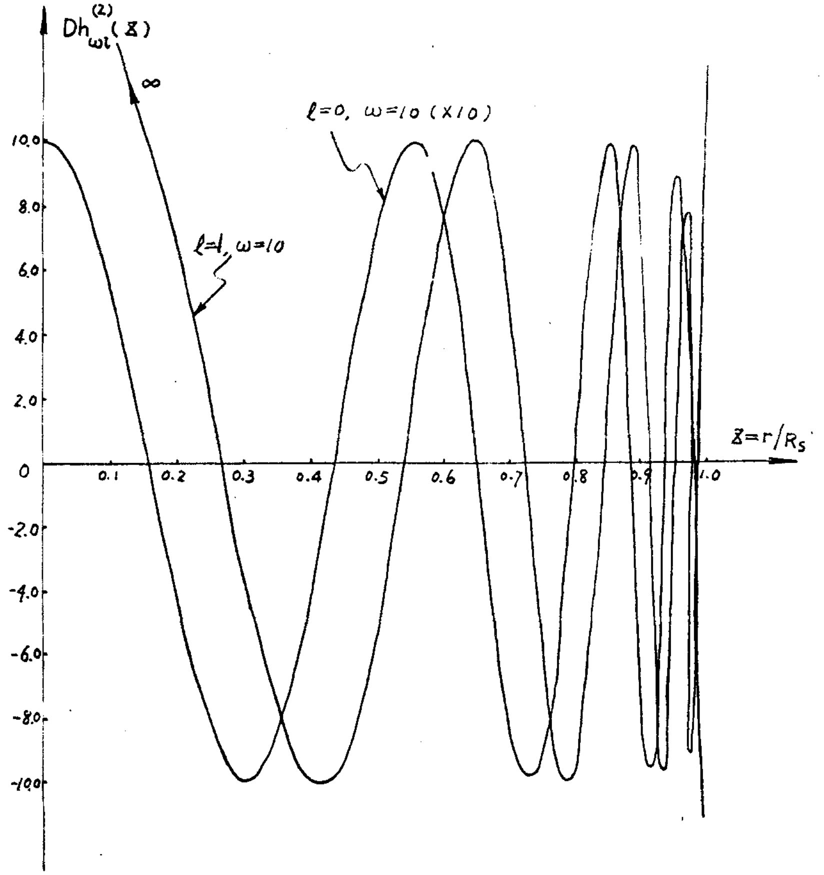

Fig. 2. Plot of $Dh^{(2)}_{\omega L}(x)$ as a function of x for $a_{-L}=1$, $p=1.50$, $\omega=10$, and $L=0$, 1. The "$\times 10$" above the curve indicates a factor of ten amplification of the longitudinal scale.

in Figure 3. The first maximum of $Dh^{(1)}_{\omega L}(x)$ for smaller ω is farther from the center. But $Dh^{(1)}_{\omega L}(x)$ for $\omega=0$ is a monotone function of x, and then the wave function of a scalar particle attenuates in an exponential way.

In the region between the event horizon and the center of the black hole, the wave function with smaller L and larger w will appear as an oscillating state with variable frequency and constant amplitude. However, the wave function with larger L and smaller w will reduce in an exponential way near the event horizon, and almost no penetration will take place at the event horizon.

In order to reveal clearly the anomalous behavior of a scalar particle in the interior neighborhood of the event horizon and in the central region

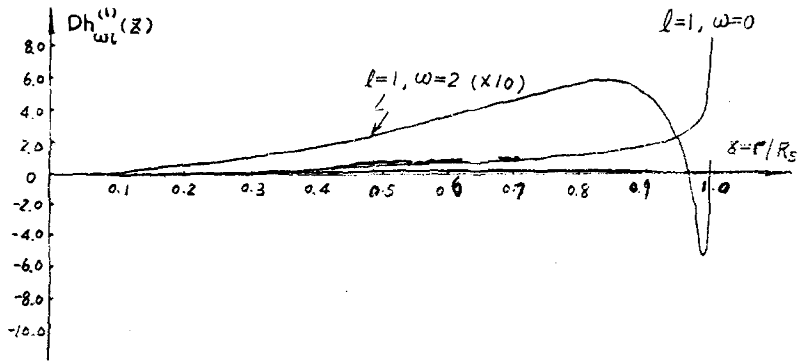

Fig. 3. Plot of $Dh^{(1)}_{\omega L}(x)$ as a function of x for $a_{L+1}=1$, $p=1.50$, $L=1$, and $\omega=0$, 2.0. The "×10" above the curve indicates a factor of ten amplification of the longitudinal scale.

of the black hole, we study the explicit approximate expressions for the radial wave function in these two regions.

4. RADIAL SOLUTION IN THE INTERIOR NEIGHBORHOOD OF THE EVENT HORIZON

In the interior neighborhood of the event horizon, the radial wave equation (5) can reduce to

$$\frac{d^2y_L(z)}{dz^2}+\left(\frac{1}{z}-\frac{7}{2R_s}\right)\frac{dy_L(z)}{dz}+\left(a+\frac{b}{z}+\frac{c}{z^2}\right)y_L(z)=0 \tag{31}$$

with

$$a=\frac{1}{4}\left[\mu^2-\frac{3L(L+1)}{R_s^2}\right],\qquad b=-\frac{1}{4}\left(2R_s\mu^2+\frac{2L(L+1)}{R_s}\right),\qquad c=\tfrac{1}{4}w^2R_s^2 \tag{32}$$

where $z=R_s-r$, $z/R_s\ll 1$, and terms of order (z^3/R_s) and its higher-order terms are neglected.

Equation (31) is a confluent hypergeometric differential equation which on integration leads to a confluent hypergeometric function in the general case and to a Bessel function (Kofinti, 1984) in the particular case.

If

$$Q=\left[\mu^2-\frac{3L(L+1)}{R_s^2}\right]\frac{4R_s^2}{49}\neq 1 \tag{33}$$

the solution of equation (31) is

$$y_L(z)=\exp\left(-\frac{i}{z}wR_s\ln z\right)\exp\left\{\frac{7}{4R_s}[1-(1+\Delta)^{1/2}]z\right\}$$
$$\times\left\{C_L\,{}_1F_1\left[\alpha,\gamma,\frac{7}{2R_s}(1+\Delta)^{1/2}z\right]\right.$$
$$\left.+D_L\exp(iwR_s)\ln z)\,{}_1F_1\left[1+\alpha-\gamma,2-\gamma,\frac{7}{2R_s}(1+\Delta)^{1/2}z\right]\right\} \tag{34}$$

where

$$q=\pm\frac{i}{2}wR_s,\qquad \Delta=\frac{4R_s^2}{49}\left[\frac{3L(L+1)}{R_s^2}-\mu^2\right],\qquad p=\frac{7}{4R_s}[1-(1+\Delta)^{1/2}]$$
$$\alpha=\frac{2pq+p+b-7q/2R_s}{(2p-7/2R_s)},\qquad \gamma=1+2q \tag{35}$$

C_L and D_L ($L=0, 1, 2, 3, \ldots$) are arbitrary constants. On the other hand, if $Q=1$, the solution of equation (31) is

$$y_L(z)=e^{(7/4R_s)z}\left\{A_LJ_\nu\left[\frac{z}{R_s}(7-2R_s^2\mu^2-2L(L+1)\right]\right.$$
$$\left.+B_LJ_{-\nu}\left[\frac{z}{R_s}(7-2R_s^2\mu^2-2L(L+1)\right]\right\} \tag{36}$$

where $\nu=+iwR_s$.

However, if we use the boundary condition in Kofinti (1984) that the waves at $r=R_s$ are pure ingoing (i.e., there is no scalar rediation from the black hole), then the constants D_L ($L=0, 1, 2, 3, \ldots$) will be zero.

The solution (34) has a phase factor which is divergent in logarithmic form at the event horizon and the corresponding radial probability current density is

$$J_r^{(L)}|_{z=0^+}=-\frac{1}{\mu}\operatorname{Im}\left[y_L^*(z)\frac{dy_L(z)}{dz}\right]\propto\frac{1}{z}\to\infty \tag{37}$$

Thus, the black hole behaves like a big trap for the scalar particle, and all particles would be absorbed into the black hole from out of the event horizon. But the radial probability density at the event horizon

$$P_r^{(L)}|_{z=0^+}=\frac{w}{\mu}|y_L(z)|^2\Big|_{z=0^+} \tag{38}$$

is finite. Equation (34) shows that the wave function with larger L (i.e., $\Delta\gg1$) has weak penetration and rapid attenuation. Hence the scalar particle with larger L is localized near the event horizon. These results are identical with those given by the exact solutions (25)-(30) and the numerical analysis.

5. RADIAL SOLUTION IN THE CENTRAL REGION OF THE BLACK HOLE

In the central region of the Schwarzschild black hole (i.e., $r \ll 1$), the terms in equation (5)

$$(2/r)(1-r^2/R_s^2)(1-2r^2/R_s^2)\,dR_L/dr$$

and

$$(1-r^2/R_s^2)\,[L(L+1)/r^2]R_L$$

become very large. In making the approximation, the small quantity (r^2/R_s^2) in metric (2) and in equation (5) is retained so as to avoid an excessive error. When $L \geq 1$, and the higher-order terms of (r^2/R_s^2) are neglected, the radial wave function can reduce to

$$r^2\frac{d^2R_L}{dr^2}+2r\frac{dR_L}{dr}+\left[\left(w^2-\mu^2-\frac{2L(L+1)}{R_s^2}\right)r^2-L(L+1)\right]R_L=0 \quad (39)$$

When

$$k^2=\left(w^2-\mu^2-\frac{2L(L+1)}{R_s^2}\right)\geq 0$$

the solution of equation (39) is

$$R_L(r)=A_L j_L(kr) \quad (40)$$

and when

$$p^2=\left(\mu^2+\frac{2L(L+1)}{R_s^2}-w^2\right)\geq 0$$

we have

$$R_L(r)=A'_L i_L(pr) \quad (41)$$

Here $j_L(kr)$ and $i_L(pr)$ stand for the spherical Bessel functions with real and imaginary variables, respectively. Clearly, either (40) or (41) satisfies $R_L(0)=0$, and consequently no scalar particle with nonzero orbital angular momentum will occur at the center of the black hole (see Figure 1).

In general, the condition $p^2 \geq 0$ is satisfied for particles with lower frequency and large orbital angular momentum. Then the wave function does not reach the minimum except at the origin. Nevertheless, under the condition $k^2 \geq 0$ we have the following recurrence relation (Abramowitz and Stegun, 1964):

$$\frac{dj_L(x)}{dx}=\frac{Lj_L(x)-xj_{L+1}(x)}{x} \quad (42)$$

for the particle with smaller L and larger w. The wave function reaches the extreme value at the position determined by

$$Lj_L(kr_m) - kr_m j_{L+1}(kr_m) = 0 \tag{43}$$

from equation (43) we obtain the position of the first maximum

$$r_m = \frac{1}{k}\left[\frac{2L(2L+5)}{L+2}\right]^{1/2} \tag{44}$$

where terms $o(r^3)$ are ignored. Only particles with lower orbital angular momentum and higher energy can penetrate into the region near the center. The distance of the first maximum away from the origin increases with L (see Figure 3).

When $L=0$, and the higher-order terms of order (r^2/R_s^2) are ignored, the radial wave equation takes the form

$$r^2\frac{d^2R_0}{dr^2} + 2r\frac{dR_0}{dr} + \left[(w^2-\mu^2)r^2 + \frac{r^4}{R_s^2}(3w^2-2\mu^2)\right]R_0 = 0 \tag{45}$$

When

$$k^2 = (w^2-\mu^2) \geq 0$$

and we put

$$R_0(r) = \left(\frac{\pi}{2kr}\right)^{1/2} u_0(kr)$$

the radial wave equation reduces to

$$r^2\frac{d^2u_0}{dr^2} + r\frac{du_0}{dr} + (k^2r^2 - \tfrac{1}{4})u_0 = 0 \tag{46}$$

Then we find

$$R_0(r) = A_0 j_0(kr) + B_0 h_0^{(1)}(kr) \tag{47}$$

where $h_0^{(1)}(kr)$ is the Hankel function. When

$$p^2 = \mu^2 - w^2 \geq 0$$

we have

$$R_0(r) = A_0'\left(\frac{\pi}{2pr}\right)^{1/2} I_{1/2}(pr) + B_0'\left(\frac{\pi}{2pr}\right)^{1/2} I_{-1/2}(pr) \tag{48}$$

where $I_{\pm 1/2}(x)$ are Bessel functions with imaginary variables.

For scalar particle with zero orbital angular momentum an infinite potential pit occurs at the center of the black hole because the solutions

(47) and (48) yield $R_L(0) \to \infty$, which corresponds to the fact that all zero orbital angular momentum particles are absorbed by the singularity at the origin.

The results obtained here are consistent with the exact solutions (25)-(30). As $r \to 0$. The exact solutions give

$$R_L(r) = \begin{cases} C_L a_{L+1} r^L / R_s^{L+1} & \text{for} \quad L \geq 1 \\ C_0 a_1 / R_s + d_0 b_0 / r & \text{for} \quad L = 0 \end{cases}$$

The above expressions accord with the spherical Bessel functions in the central region of the black hole.

6. CONCLUSION

We have found the stationary and localized states of a scalar particle and the corresponding continuous energy spectra. It can be proved that no bound state of the scalar particle exists inside the Schwarzschild black hole. In fact, if we make the substitution

$$r/R_s = \mathrm{th}(r^*/R_s), \qquad 0 \leq r^* < \infty \tag{50}$$

equation (12) reduces to the wave function equation with an independent variable r^*,

$$\frac{d^2 u_L(r^*)}{dr^{*2}} + [w^2 - U(r^*)] u_L(r^*) = 0 \tag{51}$$

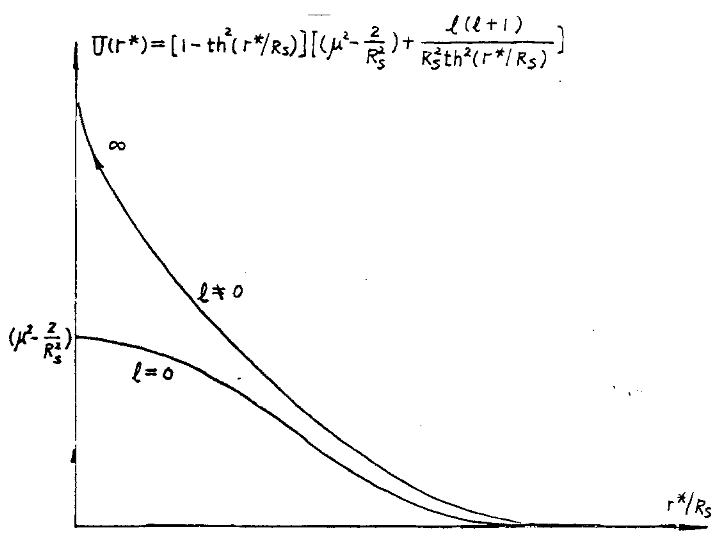

Fig. 4. Effective potential curves for $L=0$ and $L \neq 0$.

where the effective potential $U(r^*)$ is given by

$$U(r^*)=\left[1-\text{th}^2\left(\frac{r^*}{R_s}\right)\right]\left[\left(\mu^2-\frac{2}{R_s^2}\right)+\frac{L(L+1)}{R_s^2\,\text{th}^2(r^*/R_s)}\right] \tag{52}$$

As shown in Figure 4, the effective potential decreases monotonically with r^* within the range $0\leq r^*<\infty$. The effective potential does not reach the extreme value except at the event horizon $r=R_s$. This implies that no bound state of the scalar particle exists inside the Schwarzschild black hole. It should be emphasized that the bound states of the particle can emerge inside the black hole under certain conditions which we will discuss elsewhere.

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